

# 1 Diffusion

If a synapse releases a neurotransmitter, that neurotransmitter will diffuse across the synaptic cleft. In general, if you put a fluid in another fluid, the two fluids will mix through the process of diffusion. What causes this?

Various people originally thought that molecules of one kind feel a force from molecules of another kind. Einstein (good old Al) showed that diffusion is merely the result of a random walk. In other words, molecules with no preferred direction to go will follow the law of diffusion.

# 2 Diffusion as a Random Walk

Suppose you have a fluid at some temperature  $T$ . Using statistical mechanics, we can derive the velocity distribution (i.e. the percentages of molecules with velocity  $v$ ). This is all fun and good, but it's a little more information than we really need. Suffice it to say that a more rigorous analysis will turn up the same results.

For simplicity, we'll only consider motion in one dimension.

Let the average time between collisions be  $\tau$ . Let  $\epsilon$  be the distance that the average particle moves in time  $\tau$ .

Suppose a given particle  $p$  has a position  $r_p(t)$  at time  $t$ . At time  $t + \tau$ ,  $p$  will have a position

$$r_p(t + \tau) = \begin{cases} r_p(t) + \epsilon & \text{with probability } 1/2 \\ r_p(t) - \epsilon & \text{with probability } 1/2 \end{cases}$$

The expected value of the position  $r_p(t + \tau)$  *squared*, is given by:

$$\begin{aligned} \langle r_p(t + \tau)^2 \rangle &= \langle \frac{1}{2}(r_p(t) + \epsilon)^2 + \frac{1}{2}(r_p(t) - \epsilon)^2 \rangle \\ &= \langle \frac{1}{2}(r_p(t)^2 + 2r_p(t)\epsilon + \epsilon^2) + \frac{1}{2}(r_p(t)^2 - 2r_p(t)\epsilon + \epsilon^2) \rangle \\ &= \langle r_p(t)^2 + \epsilon^2 \rangle \\ &= \langle r_p(t)^2 \rangle + \langle \epsilon^2 \rangle \\ &= \langle r_p(t)^2 \rangle + \epsilon^2 \end{aligned}$$

Averaging over all particles  $p$ , we find that

$$\langle r(t + \tau)^2 \rangle = \langle r(t)^2 \rangle + \epsilon^2$$

If we set up our coordinates such that  $r(0) = 0$ , we have

$$\begin{aligned} \langle r(t)^2 \rangle &= \frac{t}{\tau} \epsilon^2 \\ &= \frac{\epsilon^2}{\tau} t \end{aligned}$$

Usually (for reasons that'll be mentioned later), this is written as

$$\langle r(t)^2 \rangle = 2Dt$$

This is the one dimensional version. In three dimensions, we have to look at the sum of the x, y and z displacements, which will give us a constant of 6 D.

Typical values of D (at room temperature) are on the order of  $10^{-9}$  m<sup>2</sup>/s. Typical diffusion distances are left as an exercise to the reader.

### 3 The Diffusion Equation

We've just looked at the question "how far do individual molecules diffuse?" Now let's turn it around. Suppose we stand at a given place. How many molecules pass by this point in a given time?

Again, we model it as a random walk.

Consider slabs of thickness  $\epsilon$  and area  $A$ . Every time step  $\tau$ ,  $1/2$  of the molecules in each slab move to the left and  $1/2$  move to the right.

We want the molecular *flux*, i.e. the number of molecules per unit area per time step.

Let  $c(x)$  be the concentration of molecules at a point  $x$ . The number of molecules passing through  $x$  is

$$\begin{aligned} & \# \text{ of molecules in slab } x - \epsilon - \# \text{ of molecules in slab } x + \epsilon \\ = & K(c(x - \epsilon)V - c(x + \epsilon)V) \\ = & K(c(x - \epsilon) - c(x + \epsilon))A\epsilon \end{aligned}$$

Where  $K$  is a constant factor meaning that some fixed fraction of the molecules in each slab have enough velocity to get to the next slab in time  $\tau$ . (The actual geometry is a little sticky.)

So the flux  $M$ , which is molecules per area per time, is given by

$$\begin{aligned} M &= \frac{\# \text{ of molecules}}{A\tau} \\ &= \frac{K(c(x - \epsilon) - c(x + \epsilon))A\epsilon}{A\tau} \\ &= \frac{K(c(x - \epsilon) - c(x + \epsilon))\epsilon}{\tau} \\ &= \frac{2K\epsilon^2}{\tau} \frac{c(x - \epsilon) - c(x + \epsilon)}{2\epsilon} \\ &= -\frac{2K\epsilon^2}{\tau} \frac{\partial c}{\partial x} \\ &= -\frac{2K\epsilon^2}{\tau} \frac{\partial c}{\partial x} \\ &= -D \frac{\partial c}{\partial x} \end{aligned}$$

(So  $K=1/4$ . We shall not comment further on  $K$ .)

This explains the value of  $2D$  in the other equation.