1 Cable Equation

We model our cable as a transmission line with a characteristic resistance R (ohms/cm), capacitance C (farads/cm), and cross resistance G (ohms/cm). (See Figure 1.)

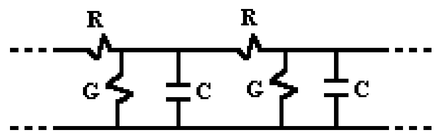

Figure 1. A Lossy Cable

Now, it is a direct result of the diagram in figure 1 that

$$\frac{\partial I}{\partial x} = -\frac{1}{C}\frac{\partial V}{\partial t} - \frac{1}{G}\frac{\partial V}{\partial x} \tag{1}$$

$$\frac{\partial V}{\partial x} = RI \tag{2}$$

Together, these imply that

$$\frac{\partial^2 V}{\partial x^2} = -\frac{R}{C}\frac{\partial V}{\partial t} - \frac{R}{G}\frac{\partial V}{\partial x}. \tag{3}$$

This is a wave equation, and has as a solution $V = V_0 e^{\kappa x + \omega t}$. Plugging this solution into the equation for V, we obtain

$$\kappa^2 V_0 = -\frac{R}{C}\omega V_0 - \frac{R}{G}\kappa V_0 \tag{4}$$

and finally that

$$\kappa = -\frac{1}{2}\frac{R}{G} \pm \frac{1}{2G}\sqrt{\frac{R}{C}}\sqrt{RC - 4\omega G^2}. \tag{5}$$

This solution tells us that if an oscillating wave of frequency ω is introduced into the cable, it will propagate down the cable with two limiting conditions.

1: If $\frac{RC}{4G^2} \geq \omega$ we have a decaying exponential as growing exponentials are unphysical

2: If $\frac{RC}{4G^2} \leq \omega$, we have a decaying oscillatory wave. This wave has a length constant of $-\frac{1}{2}\frac{R}{G}$.

Now, given this general behavior of the lossy cable (not including inductances), we will attempt to discern the behavior of the axon, viewed as a transmission cable. First, we notice that if $\omega \ll 1$ then the behavior is purely governed by a decaying exponential whose decay constant is $\frac{R}{G}$. This might the case if an axon was impaled by a pipette introducing current into the axon. In order to understand more clearly what this means, we must try to understand how R and G relate to the cell's physical characteristics.

First, we note that the axon is a cylinder. Because of this, the membrane resistance per centimeter is related to the membrane resistance per square centimeter via the relation

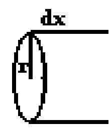

Figure 2.
Membrane Resistance

$$2\pi rg \cdot dx = G^2 dx^2 \tag{6}$$

where g gives the resistance per square centimeter. The relation for the intracellular resistance per centimeter squared is given by

$$\pi r^2 r_i = R^2 dx^2 \tag{7}$$

yeilding the final relation

$$\frac{R}{G} = \sqrt{\frac{2g}{rr_i}} \tag{8}$$

So that if we let $g' = 2g$ and $rr_i = r_i'$, then

$$\frac{R}{G} = \sqrt{\frac{g'}{r_i'}}. \tag{9}$$

Thus, for low wavelength oscillations, the wave propagates with a length constant that varies as the square root of both the membrane resistance and the internal transverse resistance. For high frequencies, the relation is halved, indicating that the potential will spread more effectively for high frequencies than for low frequencies.